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Topic: Vector Differentiation

UNIT-III

INTRODUCTION

In this chapter we study the basics of vector calculus with the help of a standard vector differential operator. Also we introduce concepts like gradient of a scalar valued function, divergence and curl of a vector valued function, discuss briefly the properties of these concepts and study the applications of the results to the evaluation of line and surface integrals in terms of multiple integrals.

2.1 GRADIENT – DIRECTIONAL DERIVATIVE

Vector differential operator

The vector differential operator ∇ (read as Del) is denoted by $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes $OX, OY \text{ and } OZ$.

It is also called Hamiltonian operator and it is neither a vector nor a scalar, but it behaves like a vector.

The gradient of a scalar function

If$\varphi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of φ is defined as $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

It is also denoted as Grad φ.

Note

(i) $\nabla\varphi$ is a vector quantity.

(ii) $\nabla\varphi = 0$ if φ is constant.

(iii) $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

(iv) $\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{\varphi^2{}_2}$ if $\varphi_2 \neq 0$

(v) $\nabla(\varphi \pm \chi) = \nabla\varphi \pm \nabla\chi$

Problems based on Gradient

Example: 2.1 Find the gradient of φ where φ is $3x^2y - y^3z^2$ at $(1, -2, 1)$.

Solution:

Given $\varphi = 3x^2y - y^3z^2$

Grad $\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

Now $\frac{\partial\varphi}{\partial x} = 6xy$, $\frac{\partial\varphi}{\partial y} = 3x^2 - 3y^2z^2$, $\frac{\partial\varphi}{\partial z} = -2y^3z$

$\therefore$ grad $\varphi = \vec{i}\,6xy + \vec{j}(3x^2 - 3y^2z^2) - \vec{k}2y^3z$

$\therefore (grad\ \varphi)_{(1,-2,\ 1)} = -12\vec{i} - 9\vec{j} + 16\vec{k}$

Example: 2.2 If $\varphi = \log(x^2 + y^2 + z^2)$ then find $\nabla\varphi$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{\imath}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{\jmath}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) = \frac{2}{r^2}\vec{r}$$

Example: 2.3 Find $\nabla(r), \nabla\left(\frac{1}{r}\right), \nabla(\log r)$ where $r = |\vec{r}|$ and $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

Solution:

Given $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$$\Rightarrow |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla(r) = \vec{\imath}\frac{\partial r}{\partial x} + \vec{\jmath}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$$= \vec{\imath}\frac{x}{r} + \vec{\jmath}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) = \frac{1}{r}\vec{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{\imath}\frac{\partial\left(\frac{1}{r}\right)}{\partial x} + \vec{\jmath}\frac{\partial\left(\frac{1}{r}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r}\right)}{\partial z}$

$$= \vec{\imath}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \vec{\jmath}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \left(-\frac{1}{r^2}\right)\left[\vec{\imath}\frac{x}{r} + \vec{\jmath}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= -\frac{1}{r^3}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) = -\frac{1}{r^3}\vec{r}$$

(iii) $\nabla(\log r) = \sum\vec{\imath}\,\frac{\partial(\log r)}{\partial x}$

$$= \sum\vec{\imath}\,\frac{1}{r}\frac{\partial r}{\partial x}$$

$$= \sum\vec{\imath}\,\frac{1}{r}\frac{x}{r}$$

$$= \sum\vec{\imath}\,\frac{x}{r^2}$$

$$= \frac{1}{r^2}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) = \frac{1}{r^2}\vec{r}$$

Example: 2.4 Prove that $\nabla(r^n) = nr^{n-2}\ \vec{r}$

Solution:

Given $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$$\nabla(r^n) = \vec{\imath}\frac{\partial r^n}{\partial x} + \vec{\jmath}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\right]$$

$$= \frac{nr^{n-1}}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = nr^{n-2}\vec{r}$$

Example: 2.5 Find $|\nabla\varphi|$ $if\ \varphi = 2xz^4 - x^2y\ at\ (2,-2,-1)$

Solution:

Given $\varphi = 2xz^4 - x^2y$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 2z^4 - 2xy,\ \ \frac{\partial\varphi}{\partial y} = -x^2,\ \ \frac{\partial\varphi}{\partial z} = 8xz^3$

$$\therefore \nabla\varphi = \vec{i}\,(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$

$$\therefore (\nabla\varphi)_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$|\nabla\varphi| = \sqrt{100 + 16 + 256} = \sqrt{372}$$

Directional Derivative (D.D) of a scalar point function

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar function φ in a given direction $\vec{a}$ is the rate of change of φ in that direction. It is given by the component of $\nabla\varphi$ in the direction of $\vec{a}$.

The directional derivative of a scalar point function in the direction of $\vec{a}$ is given by

$$\textbf{D.D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

The maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad}\ \varphi|$.

Problems based on Directional Derivative

Example: 2.6 Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1,-2,1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given $\varphi = 4xz^2 + x^2yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xyz + 4z^2) + \vec{j}\,(x^2z) + \vec{k}\,(x^2y + 8xz)$$

$$\therefore (\nabla\varphi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Given $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$$|\vec{a}| = \sqrt{4 + 1 + 4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= \left(8\vec{i} - \vec{j} - 10\vec{k}\right)\cdot\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}\ (16 + 1 + 20) = \frac{37}{3}$$

Example: 2.7 Find the directional derivative of $\varphi(x, y, z) = xy^2 + yz^3$ at the point $P(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$

Solution:

Given $\varphi = xy^2 + yz^3$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{\imath}\,(y^2) + \vec{\jmath}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$\therefore (\nabla\varphi)_{(2,-1,\ 1)} = \vec{\imath} - 3\vec{\jmath} - 3\vec{k}$$

Given $\vec{a} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \left(3\vec{\imath} + \vec{\jmath} + 3\vec{k}\right) - \left(2i - \vec{\jmath} + \vec{k}\right)$$

$$= \vec{\imath} + 2\vec{\jmath} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1 + 4 + 4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{\left(\vec{\imath} - 3\vec{\jmath} - 3\vec{k}\right)\cdot\left(\vec{\imath} + 2\vec{\jmath} + 2\vec{k}\right)}{3}$$

$$= \frac{1}{3}\ (1 - 6 - 6) = -\frac{11}{3}$$

Example: 2.8 In what direction from $(-1, 1, 2)$ is the directional derivative of $\varphi = xy^2\, z^3$ a maximum? Find also the magnitude of this maximum.

Solution:

Given $\varphi = xy^2\, z^3$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{\imath}\,(y^2z^3) + \vec{\jmath}\,(2xy\, z^3) + \vec{k}\,(3xy^2z^2)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 2)} = 8\vec{\imath} - 16\vec{\jmath} - 12\vec{k}$$

The maximum directional derivative occurs in the direction of $\nabla\varphi = 8\vec{\imath} - 16\vec{\jmath} - 12\vec{k}$.

$\therefore$ The magnitude of this maximum directional derivative

$$|\nabla\varphi| = \sqrt{64 + 256 + 144} = \sqrt{464}$$

Example: 2.9 Find the directional derivative of the scalar function $\varphi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point$(3, 1, 3)$.

Solution:

Given $\varphi = xyz$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{\imath}\,(yz) + \vec{\jmath}\,(xz) + \vec{k}\,(xy)$$

$\therefore (\nabla\varphi)_{(3,\ 1,\ 3)} = 3\vec{\imath} + 9\vec{\jmath} + 3\vec{k}$

Given surface is $z = xy \ \Rightarrow z - xy = 0$

$$\nabla\chi = \vec{\imath}\frac{\partial\chi}{\partial x} + \vec{\jmath}\frac{\partial\chi}{\partial y} + \vec{k}\frac{\partial\chi}{\partial z}$$

$$= \vec{\imath}\,(-y) + \vec{\jmath}\,(-x) + \vec{k}\,(1)$$

Let $\vec{a} = \nabla\chi_{(3,1,3)} = -\vec{\imath} - 3\vec{\jmath} + \vec{k}$

$$\Rightarrow |\vec{a}| = \sqrt{1+9+1} = \sqrt{11}$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{(3\vec{\imath}+9\vec{\jmath}+3\vec{k})\cdot(-\vec{\imath}-3\vec{\jmath}+\vec{k})}{\sqrt{11}}$$

$$= \frac{1}{\sqrt{11}}(-3-27+3) = -\frac{27}{\sqrt{11}}$$

Example: 2.10 Find the directional derivative of $\varphi = xy + yz + zx$ at $(1, 2, 0)$ in the direction of $\vec{\imath} + 2\,\vec{\jmath} + 2\vec{k}$. Find also its maximum value.

Solution:

Given $\varphi = xy + yz + zx$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{\imath}\,(y+z) + \vec{\jmath}\,(x+z) + \vec{k}\,(y+x)$$

$\therefore (\nabla\varphi)_{(1,\ 2,\ 0)} = 2\vec{\imath} + \vec{\jmath} + 3\vec{k}$

Given $\vec{a} = \vec{\imath} + 2\vec{\jmath} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{(2\vec{\imath}+\vec{\jmath}+3\vec{k})\cdot(\vec{\imath}+2\vec{\jmath}+2\vec{k})}{3}$$

$$= \frac{1}{3}(2+2+6) = \frac{10}{3}$$

Maximum value is $|\nabla\varphi| = \sqrt{4+1+9} = \sqrt{14}$

Unit normal vector to the surface

If $\varphi(x, y, z)$ be a scalar function, then $\varphi(x, y, z) = c$ represents a surface and the unit normal vector to the surface φ is given by $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$

Normal Derivative $= |\nabla\varphi|$

Problems based on unit normal vector

Example: 2.11 Find the unit normal to the surface $x^2 + y^2 = z$ at the point $(1, -2, 5)$.

Solution:

Given $\varphi = x^2 + y^2 - z$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 5)} = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\varphi| = \sqrt{4+16+1} = \sqrt{21}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i}-4\vec{j}-\vec{k}}{\sqrt{21}}$

Example: 2.12 Find the unit normal to the surface $x^2 + xy + y^2 + xyz$ at the point $(1, -2, 1)$.

Solution:

Given $\varphi = x^2 + xy + y^2 + xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x + y + yz) + \vec{j}\,(x + 2y + xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 1)} = -2\vec{i} - 2\vec{j} - 2\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i}-2\vec{j}-2\vec{k}}{2\sqrt{3}}$

$$= \frac{-1}{\sqrt{3}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

Example: 2.13 Find the normal derivative to the surface $x^2y + xz^2$ at the point $(-1, 1, 1)$.

Solution:

Given $\varphi = x^2y + xz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xy + z^2) + \vec{j}\,(x^2) + \vec{k}\,(2xz)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 1)} = -\vec{i} + \vec{j} - 2\vec{k}$$

Normal derivative $|\nabla\varphi| = \sqrt{1+1+4} = \sqrt{6}$

Example: 2.14 What is the greatest rate of increase of $\varphi = xyz^2$ at the point $(1, 0, 3)$.

Solution:

Given $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y\,z^2) + \vec{j}\,(xz^2) + \vec{k}\,(2xyz)$$

$$\therefore (\nabla\varphi)_{(1,\ 0,\ 3)} = 0\vec{i} + 9\vec{j} + 0\vec{k}$$

$\therefore$ Greatest rate of increase $|\nabla\varphi| = \sqrt{9^2} = 9$

Angle between the surfaces

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}\right]$$

Problems based on angle between two surfaces

Example: 2.15 Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, 2)$.

Solution:

Given $\varphi = x^2 + y^2 - z - 3$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi_1)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(2z)$$

$$\therefore (\nabla\varphi_2)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

$$= \frac{(4\vec{i}-2\vec{j}-\vec{k})\,(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}(6)}$$

$$= \frac{16+4-4}{\sqrt{21}(6)}$$

$$= \frac{16}{\sqrt{21}(6)} = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

Example: 2.16 Find the angle between the normals to the surfaces $x^2 = yz$ at the point $(1, 1, 1)$ *and* $(2, 4, 1)$.

Solution:

Given $\varphi = x^2 - yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(-z) + \vec{k}\,(-y)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 2\vec{i} - \vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$$

$$\therefore (\nabla\varphi_2)_{(2,\ 4,\ 1)} = 4\vec{i} - \vec{j} - 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16+1+16} = \sqrt{33}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

$$= \frac{(2\vec{i}-\vec{j}-\vec{k})\,(4\vec{i}-\vec{j}-4\vec{k})}{\sqrt{6}\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}}$$

$$= \frac{13}{\sqrt{2(3)}\sqrt{11(3)}} = \frac{13}{3\sqrt{22}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{13}{3\sqrt{22}}\right]$$

Example: 2.17 Find the angle between the surfaces $x\, log\, z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1, 1, 1)$.

Solution:

Given $\varphi_1 = y^2 - x\, log\, z - 1$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(-\log z) + \vec{j}\,(2y) + \vec{k}\left(-\frac{x}{z}\right)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 0\vec{i} + 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{0+4+1} = \sqrt{5}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2xy) + \vec{j}\,(x^2) + \vec{k}\,(1)$$

$$\therefore (\nabla\varphi_2)_{(1,\ 1,\ 1)} = 2\vec{i} + \vec{j} + \vec{k}$$

$$|\nabla\varphi_2| = \sqrt{4+1+1} = \sqrt{6}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\ |\nabla\varphi_2|}$

$$= \frac{(0\vec{i}+2\vec{j}-\vec{k})\cdot(2\vec{i}+\vec{j}+\vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{0+2-1}{\sqrt{30}}$$

$$= \frac{1}{\sqrt{30}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$$

Problems based on orthogonal surfaces

Two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

Example: 2.18 Find a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.

Solution:

Given $ax^2 - byz = (a+2)x$

Let $\varphi_1 = ax^2 - byz - (a+2)x$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$= \vec{\imath}\left(2ax - (a+2)\right) + \vec{\jmath}\,(-bz) + \vec{k}\,(-by)$

$\therefore (\nabla\,\varphi_1)_{(1,-1,\ 2)} = \vec{\imath}(a-2) + \vec{\jmath}\,(-2b) + \vec{k}(b)$

Let $\varphi_2 = 4x^2y + z^3 - 4$

$\nabla\varphi_2 = \vec{\imath}\frac{\partial\varphi_2}{\partial x} + \vec{\jmath}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{\imath}\,(8xy) + \vec{\jmath}\,(4x^2) + \vec{k}\,(3z^2)$

$\therefore (\nabla\,\varphi_2)_{(1,-1,\ 2)} = -8\vec{\imath} + 4\vec{\jmath} + 12\vec{k}$

Since the two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\,\varphi_2 = 0$

$\Rightarrow \left(\vec{\imath}(a-2) + \vec{\jmath}\,(-2b) + \vec{k}(b)\right) \cdot \left(-8\vec{\imath} + 4\vec{\jmath} + 12\vec{k}\,\right) = 0$

$\Rightarrow -8\,(a-2) - 8b + 12b = 0$

$\Rightarrow -8a + 16 - 8b + 12b = 0$

$\Rightarrow -8a + 16 + 4b = 0$

$\div$ by 4 $\Rightarrow -2a + 4 + b = 0$

$\Rightarrow 2a - b - 4 = 0 \;\; \dots (1)$

To find a and b we need another equation in a and b.

The point $(1,-1,2)$ lies in $ax^2 - byz - (a+2)x = 0$

$\therefore a - b\,(-1)\,(2) - (a+2)\,(1) = 0$

$\Rightarrow a + 2b - a - 2 = 0$

$\Rightarrow 2b - 2 = 0$

$\Rightarrow b = 1$

Substitute $b = 1$ in (1) we get

$\Rightarrow 2a - 1 - 4 = 0$

$\Rightarrow 2a - 5 = 0$

$\Rightarrow a = \frac{5}{2}$

Example: 2.19 Find the values of a and b so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$.

Solution:

Given $ax^3 - by^2z = (a+3)x^2$

Let $\varphi_1 = ax^3 - by^2z - (a+3)x^2$

$\nabla\varphi_1 = \vec{\imath}\frac{\partial\varphi_1}{\partial x} + \vec{\jmath}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$

$= \vec{\imath}\left(3ax^2 - 2x(a+3)\right) + \vec{\jmath}\,(-2byz) + \vec{k}\,(-by^2)$

$\therefore (\nabla\,\varphi_1)_{(2,-1,-3)} = \vec{\imath}(8a-12) + \vec{\jmath}\,(-6b) + \vec{k}(-b)$

Let $\varphi_2 = 4x^2y - z^3 - 11$

$\nabla\varphi_2 = \vec{\imath}\frac{\partial\varphi_2}{\partial x} + \vec{\jmath}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(-3z^2)$$

$$\therefore (\nabla \varphi_2)_{(2,-1,-3)} = -16\vec{i} + 16\vec{j} - 27\vec{k}$$

Given the two surfaces cut orthogonally if $\nabla\varphi_1 \cdot \nabla \varphi_2 = 0$

$$\Rightarrow \left(\vec{i}(8a-12) + \vec{j}\,(-6b) - \vec{k}(b)\right) \cdot \left(-16\vec{i} + 16\vec{j} - 27\vec{k}\,\right) = 0$$

$$\Rightarrow -16\,(8a-12) - 16(6b) + 27b = 0$$

$$\Rightarrow -128a + 192 - 69b = 0$$

$$\Rightarrow 128a + 69b - 192 = 0 \; \ldots \; (1)$$

To find a and b we need another equation in a and b.

The point $(2,-1,-3)$ lies in $ax^3 - by^2z - (a+3)x^2 = 0$

$$\therefore 8a - b\,(1)\,(-3) - (a+3)\,(4) = 0$$

$$\Rightarrow 4a + 3b - 12 = 0 \ldots (2)$$

Solving (1) and (2) we get, $a = -\frac{7}{3}$ & $b = \frac{64}{9}$

Equation of the tangent plane and normal to the surface

Equation of the tangent plane is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

Equation of the normal line is $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

Problems based on tangent plane

Example: 2.20 Find the equation of the tangent plane and normal line to the surface $xyz = 4$ at the point $\vec{i} + 2\vec{j} + 2\vec{k}$.

Solution:

Given $\varphi = xyz - 4$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla \varphi)_{(1,\ 2,\ 2)} = 4\vec{i} + 2\vec{j} + 2\vec{k}$$

Equation of the tangent plane at the point $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$ is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

$$\Rightarrow \left[\left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \vec{i} + 2\vec{j} + 2\vec{k}\right] \cdot \left(4\vec{i} + 2\vec{j} + 2\vec{k}\right) = 0$$

$$\Rightarrow \left[(x-1)\,\vec{i} + (y-2)\,\vec{j} + (z-2)\,\vec{k}\right] \cdot \left(4\vec{i} + 2\vec{j} + 2\vec{k}\right) = 0$$

$$\Rightarrow 4(x-1) + 2(y-2) + 2(z-2) = 0$$

$$\Rightarrow 4x - 4 + 2y - 4 + 2z - 4 = 0$$

$$\Rightarrow 4x + 2y + 2z = 12$$

$$\Rightarrow 2x + y + z = 6$$

Equation of the normal line $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\,[2\,(y-2) - 2\,(z-2)] - \vec{j}\,[2\,(x-1) - 4\,(z-2)] + \vec{k}\,[2\,(x-1) - 4\,(y-2)]$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\vec{k}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ we get

$$\Rightarrow 2\,(y-2) - 2\,(z-2) = 0$$

$$\Rightarrow (y-2) = (z-2) \quad \dots (1)$$

$$\Rightarrow 2\,(x-1) - 4\,(z-2) = 0$$

$$\Rightarrow (x-1) = 2\,(z-2)$$

$$\Rightarrow \frac{x-1}{2} = (z-2) \qquad \dots (2)$$

$$\Rightarrow 2\,(x-1) - 4\,(y-2) = 0$$

$$\Rightarrow (x-1) = 2\,(y-2)$$

$$\Rightarrow \frac{x-1}{2} = (y-2) \qquad \dots (3)$$

From $(1), (2)$ *and* (3) we get $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$

Which is the required equation of the normal line.

Exercise: 2.1

1. Find $\nabla\varphi$ if $\varphi = \frac{1}{2}\log(x^2 + y^2 + z^2)$ **Ans:** $\frac{\vec{r}}{r^2}$

2. Find the directional derivative of

   (i) $\varphi = 2xy + z^2$ at the point $(1, -1, 3)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$. **Ans:** $\frac{14}{3}$

   (ii) $\varphi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$. **Ans:** $\frac{-11}{3}$

3. Prove that the directional derivative of $\varphi = x^3y^2z$ at $(1, 2, 3)$is maximum along the direction $9\vec{i} + 3\vec{j} + \vec{k}$. Also, find the maximum directional derivative. **Ans:** $4\sqrt{91}$

4. Find the unit tangent vector to the curve $\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2 - 65)\vec{k}$ at $t = 1.$ **Ans:** $\frac{\vec{i}+2\vec{j}-\vec{k}}{\sqrt{6}}$

5. Find a unit normal to the following surfaces at the specified points.

   (i) $x^2y + 2xz = 4$ at $(2, -2, 3)$ **Ans:** $\pm\frac{1}{3}\left(\vec{i} - 2\vec{j} - 2\vec{k}\right)$

   (ii) $x^2 + y^2 = z$ at $(1, -2, 5)$ **Ans:** $\frac{1}{\sqrt{21}}\left(2\vec{i} - 4\vec{j} - \vec{k}\right)$

   (ii) $xy^3z^2 = 4$ at $(-1, -1, 2)$ **Ans:** $\frac{1}{\sqrt{11}}\left(-\vec{i} - 3\vec{j} + \vec{k}\right)$

   (iv) $x^2 + y^2 = z$ at $(1, 1, 2)$ **Ans:** $\frac{1}{3}\left(2\vec{i} + 2\vec{j} - \vec{k}\right)$

6. Find the angle between the surfaces $x^2 - y^2 - z^2 = z$ and $xy + yz - zx - 18 = 0$ at the

point $(6, 4, 3)$. **Ans:** $cos^{-1}\left[\frac{-24}{\sqrt{86}\sqrt{61}}\right]$

7. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$. **Ans:** $cos^{-1}\left[\frac{-3}{7\sqrt{6}}\right]$

8. Find the equation of the tangent plane to the surfaces $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$. **Ans:** $7x - 3y + 8z - 26 = 0$

9. Find the equation of the tangent plane to the surfaces $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$. **Ans:** $2x - y - 3z + 1 = 0$

10. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1, 1, 1)$. **Ans:** $cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$

2.2 DIVERGENCE, CURL – IRROTATIONAL AND SOLENOIDAL VECTORS

Divergence of a vector function

If $\vec{F}(x, y, z)$ is a continuously differentiable vector point function in a given region of space, then the divergence of $\vec{F}$ is defined by

$$\nabla.\vec{F} = \text{div } \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \text{where} \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Note: $\nabla.\vec{F}$ Is a scalar point function.

Solenoidal vector

A vector $\vec{F}$ is said to be solenoidal if $div\,\vec{F} = 0$ (i.e)$\nabla.\vec{F} = 0$

Curl of a vector function

If $\vec{F}(x, y, z)$ is a differentiable vector point function defines at each point (x, y, z) in some region of space, then the curl of $\vec{F}$ is defined by

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Note: $\nabla \times \vec{F}$ Is a vector point function.

Irrotational vector

A vector is said to be irrotational if Curl $\vec{F} = 0$ $(i.e)$ $\nabla \times \vec{F} = 0$

Scalar potential

If $\vec{F}$ is an irrotational vector, then there exists a scalar function φ such that $\vec{F} = \nabla\varphi$. Such a scalar function is called scalar potential of $\vec{F}$.

Problems based on Divergence and Curl of a vector

Example: 2.21 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then find div $\vec{r}$ and curl$\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now div $\vec{r} = \nabla \cdot \vec{r}$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

And curl $\vec{r} = \nabla \times \vec{r}$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}.$$

Example: 2.22 If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$ at the point (1,-1, 1).

Solution:

Given $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

(i) $\nabla.\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla.\vec{F}_{(1,-1,1)} = 1 + 2 + 6 = 9$$

(ii) $$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & 3yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial(-3yz^2)}{\partial y} - \frac{\partial(2x^2yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(-3yz^2)}{\partial x} - \frac{\partial(xy^2)}{\partial z}\right] + \vec{k}\left[\frac{\partial(2x^2yz)}{\partial x} - \frac{\partial(xy^2)}{\partial y}\right]$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0) + \vec{k}(4xyz - 2xy)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = \vec{i}(-3 + 2) + \vec{k}(-4 + 2)$$

$$= -\vec{i} - 2\vec{k}$$

Example: 2.23 If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, then find $\nabla \cdot \vec{F}$, $\nabla(\nabla \cdot \vec{F})$, $\nabla \times \vec{F}$, $\nabla \cdot (\nabla \times \vec{F})$, and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$

$$= (2x + 2z) + (-x + z) + 2z$$

$$= x + 5z$$

$\therefore \nabla \cdot \vec{F}_{(1,1,1)} = 6$

(ii) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial(z^2+x^2)}{\partial y} - \frac{\partial(xz-xy+yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(z^2+x^2)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial z}\right] + \vec{k}\left[\frac{\partial(xz-xy+yz)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial y}\right]$$

$$= -(x+y)\vec{\imath} - (2x - 2x)\vec{\jmath} + (y+z)\vec{k}$$

$\therefore \nabla \times \vec{F}_{(1,1,1)} = -2\vec{\imath} + 2\vec{k}$

(iii) $\nabla(\nabla \cdot \vec{F}) = \vec{i}\frac{\partial}{\partial x}(x + 5z) + \vec{j}\frac{\partial}{\partial y}(x + 5z) + \vec{k}\frac{\partial}{\partial z}(x + 5z)$

$$= \vec{\imath} + 5\vec{k}$$

$\therefore \nabla(\nabla \cdot \vec{F})_{(1,1,1)} = \vec{\imath} + 5\vec{k}$

(iv) $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x+y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y+z)$

$$= -1 + 0 + 1$$

$\nabla \cdot (\nabla \times \vec{F})_{(1,1,1)} = 0$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x+y) & 0 & y+z \end{vmatrix}$

$\therefore \nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{\imath} + \vec{k}$

Example: 2.24 Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

Solution:

Given $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

$$= \vec{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \vec{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz) + \vec{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$$

$$\vec{F} = \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$$

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z)$$

Curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$$= \vec{i}[-3x + 3x] - \vec{j}[-3y + 3y] + \vec{k}[-3z + 3z]$$

$= \vec{0}$

Example: 2.25 Find div(grad φ) and curl(grad φ) at (1,1,1) for $\varphi = x^2y^3z^4$

Solution:

Given $\varphi = x^2y^3z^4$

$$\text{grad } \varphi = \nabla \varphi = \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$$

$$= \vec{i}(2xy^3z^4) + \vec{j}(x^2 3y^2z^4) + \vec{k}(x^2y^3 4z^3)$$

$$\text{Div(grad } \varphi) = \nabla \cdot (\text{grad } \varphi)$$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(x^2 3y^2z^4) + \frac{\partial}{\partial z}(x^2y^3 4z^3)$$

$$= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^4$$

$$\therefore \text{Div(grad } \varphi)_{(1,1,1)} = 2 + 6 + 12 = 20$$

$$\text{Curl(grad } \varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & x^2 3y^2z^4 & x^2y^3 4z^3 \end{vmatrix}$$

$$= \vec{i}(12x^2y^2z^3 - 12x^2y^2z^3) - \vec{j}(8xy^3z^3 - 8xy^3z^3) + \vec{k}(6xy^2z^4 - 6xy^2z^4)$$

$$= \vec{0}$$

$$\therefore \text{Curl grad}\varphi_{(1,1,1)} = \vec{0}$$

Vector Identities

1) $\nabla \cdot (\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$
2) $\nabla \times (\varphi \vec{F}) = \varphi(\nabla \times \vec{F}) + (\nabla\varphi) \times \vec{F}$
3) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
4) $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$
5) $\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) - (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) - (\vec{B} \cdot \nabla)\vec{A}$
6) $\nabla \cdot (\nabla\varphi) = \vec{0}$
7) $\nabla \cdot (\nabla \times \vec{F}) = 0$
8) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$
9) $\nabla \cdot \nabla\varphi = (\nabla \cdot \nabla)\varphi = \nabla^2\varphi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a laplacian operator

1) If φ is a scalar point function, $\vec{F}$ is a vector point function, then $\nabla . (\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$

Proof:

$$\nabla \cdot (\varphi \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \frac{\partial}{\partial x}(\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \left(\varphi\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right)$$

$$= \varphi\left(\sum \vec{i}\cdot\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right) + \vec{F}\cdot\left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right)$$

$$\therefore \nabla\cdot(\varphi\,\vec{F}) = \varphi\left(\nabla\cdot\vec{F}\right) + \vec{F}\cdot\nabla\varphi$$

2) If φ is a scalar point fuction, $\vec{F}$ is a vector point function, then $\nabla\times(\varphi\,\vec{F}) = \varphi\left(\nabla\times\vec{F}\right) + (\nabla\varphi)\times\vec{F}$

Proof:

$$\nabla\times(\varphi\,\vec{F}) = \sum \vec{i}\times\frac{\partial}{\partial x}(\varphi\,\vec{F})$$

$$=\sum \vec{i}\times\left[\varphi\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right]$$

$$=\sum \vec{i}\times\left(\frac{\partial \varphi}{\partial x}\vec{F} + \varphi\frac{\partial \vec{F}}{\partial x}\right)$$

$$=\left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right)\times\vec{F} + \varphi\left[\sum \vec{i}\times\frac{\partial \vec{F}}{\partial x}\right]$$

$$\therefore \nabla\times(\varphi\,\vec{F}) = \nabla\varphi\times\vec{F} + \varphi(\nabla\times\vec{F})$$

3) If $\vec{A}$ and $\vec{B}$ are vector point functions, then $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

Proof:

$$\nabla.(\vec{A}\times\vec{B}) = \sum \vec{i}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$=\sum \vec{i}\cdot\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)$$

$$=\sum \vec{i}\cdot\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right) + \sum \vec{i}\cdot\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)$$

$$= -\left(\sum \vec{i}\times\frac{\partial \vec{B}}{\partial x}\right)\cdot\vec{A} + \left(\sum \vec{i}\times\frac{\partial \vec{A}}{\partial x}\right).\vec{B}$$

$$= -\left(\nabla\times\vec{B}\right).\vec{A} + (\nabla\times\vec{A})\cdot\vec{B}$$

$$\therefore \nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \qquad \left[\because \left(\nabla\times\vec{A}\right)\cdot\vec{B} = \vec{B}\cdot(\nabla\times\vec{A})\right]$$

(4) If $\vec{A}$ and $\vec{B}$ are vector point functions, then

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + \left(\vec{B}\cdot\nabla\right)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$$

Proof:

$$\nabla\times(\vec{A}\times\vec{B}) = \sum \vec{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$=\sum \vec{i}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)$$

$$=\sum \vec{i}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right) + \sum \vec{i}\times\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)$$

We know that $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = (\vec{a}\cdot\vec{c})\,\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$$\nabla\times(\vec{A}\times\vec{B}) = \sum\left[\left(\vec{i}\cdot\vec{B}\right)\frac{\partial \vec{A}}{\partial x} - \left(\vec{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right] + \sum\left[\left(\vec{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\vec{i}\cdot\vec{A}\right)\frac{\partial \vec{B}}{\partial x}\right]$$

$$=\left(\sum \vec{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \sum\left(\vec{B}\cdot\vec{i}\right)\frac{\partial \vec{A}}{\partial x} - \sum\left(\vec{A}\cdot\vec{i}\right)\frac{\partial \vec{B}}{\partial x}$$

$$=\left(\sum \vec{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \left(\vec{B}\cdot\sum \vec{i}\frac{\partial}{\partial x}\right)\vec{A} - \left(\vec{A}\cdot\sum \vec{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$\therefore \nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + \left(\vec{B}\cdot\nabla\right)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$$

(5) If $\vec{A}$ and $\vec{B}$ are vector point functions, then

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

Proof:

$$\nabla(\vec{A}\cdot\vec{B}) = \sum \vec{i}\,\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$$

$$= \sum \vec{i}\left(\frac{\partial \vec{A}}{\partial x}\cdot\vec{B} + \vec{A}\cdot\frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum \vec{i}\left(\vec{B}\cdot\frac{\partial \vec{A}}{\partial x}\right) + \sum \vec{i}\left(\vec{A}\cdot\frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum\left(\vec{B}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{i} + \sum\left(\vec{A}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{i} \quad \dots (1)$$

We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

$\therefore (\vec{a}\cdot\vec{b})\,\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$

Consider $\sum\left(\vec{B}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{i} = \sum\left[(\vec{B}\cdot\vec{i})\frac{\partial \vec{A}}{\partial x} - \vec{B}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{i}\right)\right]$

$$= \sum\left(\vec{B}\cdot\vec{i}\,\frac{\partial}{\partial x}\right)\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\times\frac{\partial \vec{A}}{\partial x}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\,\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\,\frac{\partial}{\partial x}\times\vec{A}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\,\vec{A} + \vec{B}\times(\nabla\times\vec{A}) \quad \dots (2)$$

In (2) interchanging $\vec{A}$ and $\vec{B}$ we get,

$$\sum\left(\vec{A}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{i} = (\vec{A}\cdot\nabla)\,\vec{B} + \vec{A}\times(\nabla\times\vec{B}) \quad \dots (3)$$

Substitute in equation (1)

$$(1) \Rightarrow \nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\,\vec{A} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\,\vec{B} + \vec{A}\times(\nabla\times\vec{B})$$

(6) If φ is a scalar point function, then $\nabla\times(\nabla\varphi) = \vec{0}$.

(or)

Prove that $curl(grad\ \varphi) = 0$.

Solution:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right]$$

$$= \sum \vec{i}\,(\vec{0}) = \vec{0}$$

(7) If $\vec{F}$ is a vector point function, then $\nabla\cdot(\nabla\times\vec{F}) = 0$.

(or)

Prove that $div(curl\,\vec{F}) = 0$.

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot$$

$$\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0$$

(8) If is a vector point function, then $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

(or)

Prove that $curl\,(curl\,\vec{F}) = grad\,(div\,\vec{F}) - \nabla^2\vec{F}$

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times (\nabla \times \vec{F}) = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

And $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

L.H.S $\quad \nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_3}{\partial z \partial x} + \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial z^2}\right]$$

$$+\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right]$$

R.H.S $\quad \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}\right] + \vec{j}\left[\frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}\right] + \vec{k}\left[\frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial z^2}\right] +$$

$$\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right]$$

L.H.S = R.H.S

$$\therefore \nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla^2 \vec{F}$$

(9) $\nabla \cdot (\nabla\varphi) = (\nabla \cdot \nabla)\,\varphi = \nabla^2\varphi$

Proof:

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\therefore\ \nabla \cdot (\nabla\varphi) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot (\nabla\varphi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = \nabla^2\varphi$$

Example: 2.26 Find (i) $\nabla \cdot \vec{r}$ (ii) $\nabla \times \vec{r}$

Solution:

Let $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

(i) $\nabla \cdot \vec{r} = \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{\imath}(0) + \vec{\jmath}\,(0) + \vec{k}\,(0) = \vec{0}$$

Example: 2.27 Find $\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right)$ where $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

Solution:

$$\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right) = \nabla \cdot \left[\frac{1}{r}\left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)\right]$$

$$= \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\,\vec{\imath} + \frac{y}{r}\,\vec{\jmath} + \frac{z}{r}\,\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum\left[\frac{1}{r}(1) + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right]$$

$$= \sum\left[\frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right)\right] \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

\`

$$= \sum\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \frac{3}{r} - \frac{1}{r^3}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} \qquad \because r^2 = \left(x^2 + y^2 + z^2\right)$$

$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$

Example: 2.28 If is a constant vector and is the position vector of any point, prove that

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = 0$ (ii) $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$\vec{a} = a_1\,\vec{i} + a_2\,\vec{j} + a_3\,\vec{k}$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$= \vec{i}(a_2 z - a_3 y) - \vec{j}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x)$

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(-a_1 z + a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$

$= 0 + 0 + 0 = 0$

(ii) $$\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_3 x \end{vmatrix}$$

`$= \vec{i}(a_1 + a_1) - \vec{j}(-a_2 - a_2) + \vec{k}(a_3 + a_3)$

$= 2a_1\vec{i} + 2a_2\vec{j} + 2a_3\vec{k}$

$= 2(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = 2\vec{a}$

Example: 2.29 Prove that $curl(f(r)\vec{r}) = \vec{0}$

Solution:

Let $f(r)\vec{r} = f(r)[x\,\vec{i} + y\,\vec{j} + z\,\vec{k}]$

$= xf(r)\vec{i} + yf(r)\vec{j} + zf(r)\vec{k}$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$= \sum \vec{i}\left[zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}\right]$

$= \sum \vec{i}\left[zf'(r)\left(\frac{y}{r}\right) - yf'(r)\left(\frac{z}{r}\right)\right]$

$= \sum \vec{i}\left[\frac{zy}{r}f'(r) - \frac{zy}{r}f'(r)\right]$

$= \sum \vec{i}\,(0)$

$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$

Example: 2.30 Prove that $curl[\varphi\,\nabla\varphi] = \vec{0}$

(or)

Prove that $\nabla \times [\varphi\,\nabla\varphi] = \vec{0}$

Solution:

$$\varphi\nabla\varphi = \varphi\left[\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right]$$

$$= \vec{i}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \vec{j}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \vec{k}\left(\varphi\frac{\partial\varphi}{\partial z}\right)$$

$$\nabla \times (\varphi\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial\varphi}{\partial x} & \varphi\frac{\partial\varphi}{\partial y} & \varphi\frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right]$$

$$= \sum \vec{i}\left[\varphi\frac{\partial^2\varphi}{\partial y\partial z} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z}\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

Example: 2.31 If $\vec{\omega}$ is a constant vector and $\vec{v} = \vec{\omega} \times \vec{r}$, then prove that $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$.

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$$\vec{\omega} = \omega_1\,\vec{i} + \omega_2\,\vec{j} + \omega_3\,\vec{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(\omega_2 z - \omega_3 y) - \vec{j}(\omega_1 z - \omega_3 x) + \vec{k}(\omega_1 y - \omega_2 x)$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \vec{i}(\omega_1 + \omega_1) - \vec{j}(-\omega_2 - \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1\vec{i} + 2\omega_2\vec{j} + 2\omega_3\vec{k}$$

$$= 2\left(\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}\right) = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$$

Problems based on solenoidal vector and irrotational vector and scalar potential

Example: 2.32 Prove that the vector $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.33 Show that the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.34 If $\vec{F} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + \lambda z)\vec{k}$ is solenoidal, then find the value of λ.

Solution:

Given $\vec{F}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + \lambda z) = 0$$

$$\Rightarrow 1 + 1 + \lambda = 0$$

$$\therefore\ \lambda = -2$$

Example: 2.35 Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

Solution:

Given $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$$\Rightarrow 3 + a + 2 = 0$$

$$\therefore\ a = -5$$

Example: 2.36 Show that the vector $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution:

Given $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

To prove $curl\ \vec{F} = 0$

$$(i.e) To\ prove\ \nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - 3z^2 & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}(-1 + 1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

Example: 2.37 Find the constants a, b, c so that the vectors is irrotational

$\vec{F} = (x+2y+az)\vec{i} + (bx+3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ **.**

Solution:

Given $\vec{F} = (x+2y+az)\vec{i} + (bx+3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ **is irrotational.**

$$(ie)\nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx+3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = \vec{0}$$

$$\Rightarrow c+1=0 \ ; \qquad 4-a=0 \ ; \qquad b-2=0$$

$$\Rightarrow c = -1 \ ; \qquad 4=a \ ; \qquad b=2$$

Example: 2.38 Prove that $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ **is irrotational and find** φ **such that** $\vec{F} = \nabla\varphi$**.**

Solution:

Given $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x-6x)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = 6xy+z^3$; $\qquad \frac{\partial\varphi}{\partial y} = 3x^2-z$; $\qquad \frac{\partial\varphi}{\partial z} = 3xz^2-y$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = 3x^2y + xz^3 + f_1(y,z)$

$\varphi = 3x^2y - yz + f_2(x,z)$

$\varphi = xz^3 - yz + f_3(x,y)$

$\therefore \varphi = 3x^2y + xz^3 - yz + c$ where c is constant.

Example: 2.39 Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin z - 4)\vec{j} + (3xz^2)\vec{k}$ **is irrotational and find its scalar potential.**

Solution:

Given $\vec{F} = (y^2 cos\, x + z^3)\vec{\imath} + (2y\, sinz - 4)\vec{\jmath} + (3xz^2)\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 cos\, x + z^3 & 2y\, sinz - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{\imath}(0-0) - \vec{\jmath}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2ycos\, x)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = y^2\cos x + z^3$; $\quad \frac{\partial\varphi}{\partial y} = 2ysin\, x - 4$; $\quad \frac{\partial\varphi}{\partial z} = 3xz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = y^2 \sin x + z^3 x + f_1(y,z)$

$\varphi = y^2 \sin x - 4y + f_2(x,z)$

$\varphi = xz^3 + f_3(x,y)$

$\therefore \varphi = y^2 \sin x + z^3 x - 4y + c$ is scalar potential.

Example: 2.40 Prove that $\vec{F} = (2x+yz)\vec{\imath} + (4y+zx)\vec{\jmath} + (6z-xy)\vec{k}$ is solenoidal as well as irrotational also find the scalar potential of $\vec{F}$.

Solution:

Given $\vec{F} = (2x+yz)\vec{\imath} + (4y+zx)\vec{\jmath} + (6z-xy)\vec{k}$

(i) To prove $\vec{F}$ is solenoidal.

$(ie)\ To\ prove\ \nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x+yz) + \frac{\partial}{\partial y}(4y+zx) + \frac{\partial}{\partial z}(-6z+xy)$$

$$= 2 + 4 - 6 = 0$$

$\therefore$ $\vec{F}$ is solenoidal.

(ii) To prove $\vec{F}$ is irrotational.

$(ie)\ To\ prove\ \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 4y+zx & -6z+xy \end{vmatrix}$$

$$= \vec{\imath}(x-x) - \vec{\jmath}(y-y) + \vec{k}(z-z)$$

$$= \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

(iii) To find φ such that $\vec{F} = \nabla\varphi$.

$$(2x+yz)\vec{\imath} + (4y+zx)\vec{\jmath} + (6z-xy)\vec{k} = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = 2x+yz;\qquad \frac{\partial\varphi}{\partial y} = 4y+zx;\qquad \frac{\partial\varphi}{\partial z} = -6z+xy$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2 + xyz + f_1(y,z)$

$\varphi = 2y^2 + xyz + f_2(x,z)$

$\varphi = -3z^2 + xyz + f_3(x,y)$

$\therefore \varphi = x^2 + 2y^2 - 3z^2 + xyz + c$ where c is a constant.

$\therefore\ \varphi$ is a scalar potential of $\vec{F}$.

Example: 2.41 If $\nabla\varphi = 2xyz^3\vec{\imath} + x^2z^3\vec{\jmath} + 3x^2yz^2\vec{k}$ find φ if $\varphi(-1,2,2) = 4$

Solution:

Given $\nabla\varphi = 2xyz^3\vec{\imath} + x^2z^3\vec{\jmath} + 3x^2yz^2\vec{k}$ ... (1)

We know that $\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$ ... (2)

Comparing (1) and (2)

$\frac{\partial\varphi}{\partial x} = 2xyz^3;\qquad \frac{\partial\varphi}{\partial y} = x^2z^3;\qquad \frac{\partial\varphi}{\partial z} = 3x^2yz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2yz^3 + f_1(y,z)$

$\varphi = x^2yz^3 + f_2(x,z)$

$\varphi = x^2yz^3 + f_3(x,y)$

$\therefore \varphi = x^2yz^3 + c$ where c is a constant.

Given $\varphi(-1,2,2) = 4$

$\Rightarrow 16 + c = 4$

$\Rightarrow c = -12$

$\therefore\ \varphi = x^2yz^3 - 12$

Example: 2.42 If $\vec{A}$ and $\vec{B}$ are irrotational, then prove that $\vec{A} \times \vec{B}$ is solenoidal.

Solution:

Given $\vec{A}$ and $\vec{B}$ are irrotational.

$(ie)\nabla\times\vec{A} = 0$ and $\nabla\times\vec{B} = 0$

We know that $\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - (\nabla\times\vec{B})\cdot\vec{A}$

$$= 0\cdot\vec{A} - 0\cdot\vec{B}$$

$= 0$

Hence $\vec{A} \times \vec{B}$ is solenoidal.

Example: 2.43 if $\vec{A}$ is a constant vector, then prove that (i) $div\,\vec{A} = 0$ and (ii) $curl\,\vec{A} = 0$

Solution:

Let $\vec{A} = A_1\vec{\imath} + A_2\vec{\jmath} + A_3\vec{k}$

$\frac{\partial A_1}{\partial x} = 0; \quad \frac{\partial A_2}{\partial y} = 0; \quad \frac{\partial A_3}{\partial z} = 0$

(i) $\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

$= 0 + 0 + 0 = 0$

Hence $div\,\vec{A} = 0$.

(ii) $\nabla \times \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$= \vec{\imath}(0-0) - \vec{\jmath}(0-0) + \vec{k}(0-0)$

$= \vec{0}$

$\therefore\ curl\,\vec{F} = \vec{0}$

Example: 2.44 If φ and χ are differentiable scalar fields, prove $\nabla\varphi \times \nabla\chi$ is solenoidal.

Solution:

Consider $\nabla \cdot (\nabla\varphi \times \nabla\chi)$

$= \nabla\chi \cdot \nabla \times (\nabla\varphi) - \nabla\varphi \cdot [\nabla \times (\nabla\chi)]$ $\quad [\because \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})]$

$= \nabla\chi \cdot 0 - \nabla\varphi \cdot 0$

$= 0$

$\therefore \nabla\varphi \times \nabla\chi$ is solenoidal.

Example: 2.45 Find $f(r)$ if the vector $f(r)\vec{r}$ is both solenoidal and irrotational.

Solution:

(i) Given $f(r)\vec{r}$ is solenoidal.

$\therefore\ \nabla \cdot (f(r)\vec{r}) = 0$

We know that $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$\therefore\ f(r)\vec{r} = f(r)\,x\vec{\imath} + f(r)\,y\vec{\jmath} + f(r)\,z\vec{k}$

Now $\nabla \cdot (f(r)\vec{r}) = 0$

$\Rightarrow \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(f(r)\,x\,\vec{\imath} + f(r)\,y\,\vec{\jmath} + f(r)\,z\,\vec{k}\right) = 0$

$\Rightarrow \frac{\partial}{\partial x}\,(f(r)\,x) + \frac{\partial}{\partial y}\,(f(r)\,y) + \frac{\partial}{\partial z}(f(r)\,z) = 0$

$\Rightarrow \sum \frac{\partial}{\partial x}(f(r)\,x) = 0$

$$\Rightarrow \sum \left[f(r) \cdot 1 + x\, f'(r) \frac{\partial r}{\partial x} \right] = 0$$

$$\Rightarrow \sum \left[f(r) + x\, f'(r) \frac{x}{r} \right] = 0$$

$$\Rightarrow \sum \left[f(r) + \frac{x^2}{r} f'(r) \right] = 0$$

$$\Rightarrow 3f(r) + f'(r) \left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right] = 0$$

$$\Rightarrow 3f(r) + \frac{f'(r)}{r} [r^2] = 0 \qquad [\because\ x^2 + y^2 + z^2 = r^2]$$

$$\Rightarrow 3f(r) + f'(r)\, r = 0$$

$$\Rightarrow f'(r)\, r = -3f(r)$$

$$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$$

Integrating with respect to r, we get

$$\Rightarrow \int \frac{f'(r)}{f(r)}\, dr = \int \frac{-3}{r}\, dr$$

$$\Rightarrow \log f(r) = -3 \log r + \log c$$

$$= \log r^{-3} + \log c$$

$$= \log \left(\frac{1}{r^3} \right) + \log c$$

$$= \log \left(\frac{c}{r^3} \right)$$

$$\therefore f(r) = \frac{c}{r^3}$$

(ii) Given $f(r)\vec{r}$ is irrotational.

$$\nabla \times f(r)\vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{\imath} \left[z \frac{\partial}{\partial y} f(r) - y \frac{\partial}{\partial z} f(r) \right]$$

$$= \sum \vec{\imath} \left[zf'(r) \frac{\partial r}{\partial y} - y\, f'(r) \frac{\partial r}{\partial z} \right]$$

$$= \sum \vec{\imath} \left[zf'(r) \frac{y}{r} - y\, f'(r) \frac{z}{r} \right]$$

$$= \sum \vec{\imath} f'(r) \left[\frac{zy}{r} - \frac{zy}{r} \right]$$

$$= \vec{0} \text{ for all } f(r)$$

Example: 2.46 Prove that $r^n\vec{r}$ is irrotational for every n and solenoidal only for $n = -3$.

Solution:

We know that $\vec{r} = x\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

$$\therefore\ r^n\vec{r} = r^n\, x\vec{\imath} + r^n\, y\,\vec{\jmath} + r^n\, z\,\vec{k}$$

(i) To prove $r^n\vec{r}$ is irrotational.

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n\, x & r^n\, y & r^n\, z \end{vmatrix}$$

$$= \sum \vec{i} \left[z\, nr^{n-1} \frac{\partial r}{\partial y} - y\, nr^{n-1} \frac{\partial r}{\partial z} \right]$$

$$= \sum \vec{i} \left[z\, nr^{n-1} \frac{y}{r} - y\, nr^{n-1} \frac{z}{r} \right]$$

$$= \sum \vec{i} \left[nr^{n-1} \frac{zy}{r} - nr^{n-1} \frac{zy}{r} \right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

$\therefore$ $r^n \vec{r}$ is irrotational for every n.

(ii) To prove $r^n \vec{r}$ is solenoidal.

$$\nabla \cdot (r^n \vec{r}) = \nabla \cdot \left(r^n\, x\vec{i} + r^n\, y\,\vec{j} + r^n\, z\,\vec{k} \right)$$

$$= \sum \frac{\partial}{\partial x} (r^n\, x)$$

$$= \sum \left[r^n\,(1) + x n r^{n-1} \frac{\partial r}{\partial x} \right]$$

$$= \sum \left[r^n + x n r^{n-1} \frac{x}{r} \right]$$

$$= \sum [r^n + x^2 n r^{n-2}]$$

$$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{n-2}(r^2)$$

$$= 3r^n + nr^n$$

$$= r^n(3 + n)$$

When $n = -3$, we get $\nabla \cdot (r^n \vec{r}) = 0$

$\therefore$ $r^n \vec{r}$ is solenoidal only if $n = -3$.

Problems based on Laplace operator

Example: 2.47 Find $\nabla^2(\log r)$

Solution:

$$\nabla^2(\log r) = \sum \frac{\partial^2}{\partial x^2} (\log r)$$

$$= \sum \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial r}{\partial x} \right)$$

$$= \sum \frac{\partial}{\partial x} \left(\frac{1}{r^2}\, x \right)$$

$$= \sum \left[\frac{1}{r^2}(1) + x \left(-\frac{2}{r^3} \right) \frac{\partial r}{\partial x} \right]$$

$$= \sum \left[\frac{1}{r^2} - x \left(\frac{2}{r^3} \right) \frac{x}{r} \right]$$

$$= \sum \left[\frac{1}{r^2} - \frac{2x^2}{r^4} \right]$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(x^2 + y^2 + z^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(r^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

Example: 2.48 Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$and hence deduce $\nabla^2\left(\frac{1}{r}\right)$.

(or)

Prove that $div\,(grad\, r^n) = n(n+1)\, r^{n-2}$

Solution:

Let $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$; $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n) = \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}[n\, x r^{n-2}]$$

$$= \sum n\left[x(n-2)r^{n-3} \frac{\partial r}{\partial x} + r^{n-2}\,(1)\right]$$

$$= \sum n\left[x(n-2)r^{n-3} \frac{x}{r} + r^{n-2}\right]$$

$$= \sum[n[(n-2)r^{n-4}\, x^2 + r^{n-2}]]$$

$$= \sum[n(n-2)r^{n-4}\, x^2 + n\, r^{n-2}]$$

$$= n(n-2)r^{n-4}\,(x^2 + y^2 + z^2) + 3\, n\, r^{n-2}$$

$$= n(n-2)r^{n-4}\, r^2 + 3\, n\, r^{n-2}$$

$$= n(n-2)r^{n-2} + 3\, n\, r^{n-2}$$

$$= n\, r^{n-2}\,(n-2+3)$$

$$= n\, r^{n-2}\,(n+1) \qquad \ldots (1)$$

(ii) $\nabla^2\left(\frac{1}{r}\right) = \nabla^2(r^{-1})$

$$= (-1)\,(-1+1)\, r^{-1-2} \quad \text{by (1)}$$

$$= (-1)\,(0)\, r^{-3} = 0$$

Example: 2.49 Prove that $\nabla^2(r^n\vec{r}) = n\,(n+3)r^{n-2}\vec{r}$

Solution:

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Hence $\frac{\partial \vec{r}}{\partial x} = \vec{i}$; $\quad \frac{\partial \vec{r}}{\partial y} = \vec{j}$; $\quad \frac{\partial \vec{r}}{\partial z} = \vec{k}$

Also $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r};\quad \frac{\partial r}{\partial y} = \frac{y}{r};\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n\vec{r}) = \sum \frac{\partial^2}{\partial x^2}(r^n\vec{r})$$

$$= \sum \frac{\partial}{\partial x}\left[r^n \frac{\partial \vec{r}}{\partial x} + n\, r^{n-1} \frac{\partial r}{\partial x}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\, \vec{i} + n\, r^{n-1} \frac{x}{r}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\, \vec{i} + n\, r^{n-2}\, x\, \vec{r}\right]$$

$$= \sum\left[n\, r^{n-1} \frac{\partial r}{\partial x}\vec{i} + n\left[r^{n-2}x\left(\frac{\partial \vec{r}}{\partial x}\right) + r^{n-2}(1)\vec{r} + \left[(n-2)\, r^{n-3} \frac{\partial r}{\partial x}\right] x\vec{r}\right]\right]$$

$$= \sum\left[n\, r^{n-1} \frac{x}{r}\vec{i} + n\, r^{n-2}\, x\vec{i} + n\, r^{n-2}\vec{r} + n(n-2)\, r^{n-3} \frac{x}{r}\, x\vec{r}\right]$$

$$= \sum[n\, r^{n-2}x\vec{i} + n\, r^{n-2}\, x\vec{i} + n\, r^{n-2}\vec{r} + n(n-2)\, r^{n-4}\, x^2\vec{r}]$$

$$= n\, r^{n-2}\left(x\vec{i} + y\,\vec{j} + z\,\vec{k}\right) + n\, r^{n-2}\left(x\vec{i} + y\,\vec{j} + z\,\vec{k}\right) + 3n\, r^{n-2}\vec{r}$$
$$n(n-2)\, r^{n-4}\vec{r}(x^2 + y^2 + z^2)$$

$$= n\, r^{n-2}\vec{r} + n\, r^{n-2}\vec{r} + 3n\, r^{n-2}\vec{r} + n(n-2)r^{n-4}\vec{r}r^2$$

$$= 5n\, r^{n-2}\vec{r} + n(n-2)r^{n-2}\vec{r}$$

$$= n\, r^{n-2}\vec{r}(5 + n - 2)$$

$$= n\, r^{n-2}\vec{r}(n+3)$$

$$= n(n+3)r^{n-2}\vec{r}$$

Example: 2.50 Prove that $\nabla^2 \boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{f}''(\boldsymbol{r}) + \left(\frac{2}{r}\right)\boldsymbol{f}'(\boldsymbol{r})$

Solution:

$$\nabla^2 f(r) = \sum \frac{\partial^2}{\partial x^2} f(r)$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\, \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\, \frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)x\, \frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\left[\frac{-1}{r^2}\frac{\partial r}{\partial x}\right] + f'(r)\,(1)\,\frac{1}{r} + f''(r)\,\frac{\partial r}{\partial x}\, x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\, \frac{-1}{r^2}\frac{x}{r} + f'(r)\,\frac{1}{r} + f''(r)\,\frac{x}{r}\, x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)\,\frac{-1}{r^3}x^2 + f'(r)\,\frac{1}{r} + f''(r)\,\frac{1}{r^2}\, x^2\right]$$

$$= f'(r)\,\frac{-1}{r^3}(x^2 + y^2 + z^2) + \frac{3}{r}f'(r) + f''(r)\,\frac{1}{r^2}\,(x^2 + y^2 + z^2)$$

$$= -f'(r)\,\frac{1}{r^3}(r^2) + \frac{3}{r}f'(r) + f''(r)\,\frac{1}{r^2}\,(r^2)$$

$$= -f'(r)\,\frac{1}{r} + \frac{3}{r}f'(r) + f''(r)$$

$$= f''(r) + \frac{2}{r}f'(r)$$